USING THE RANDOM ITERATION ALGORITHM TO CREATE FRACTALS

UNIVERSITY OF MARYLAND DIRECTED READING PROGRAM FALL 2015

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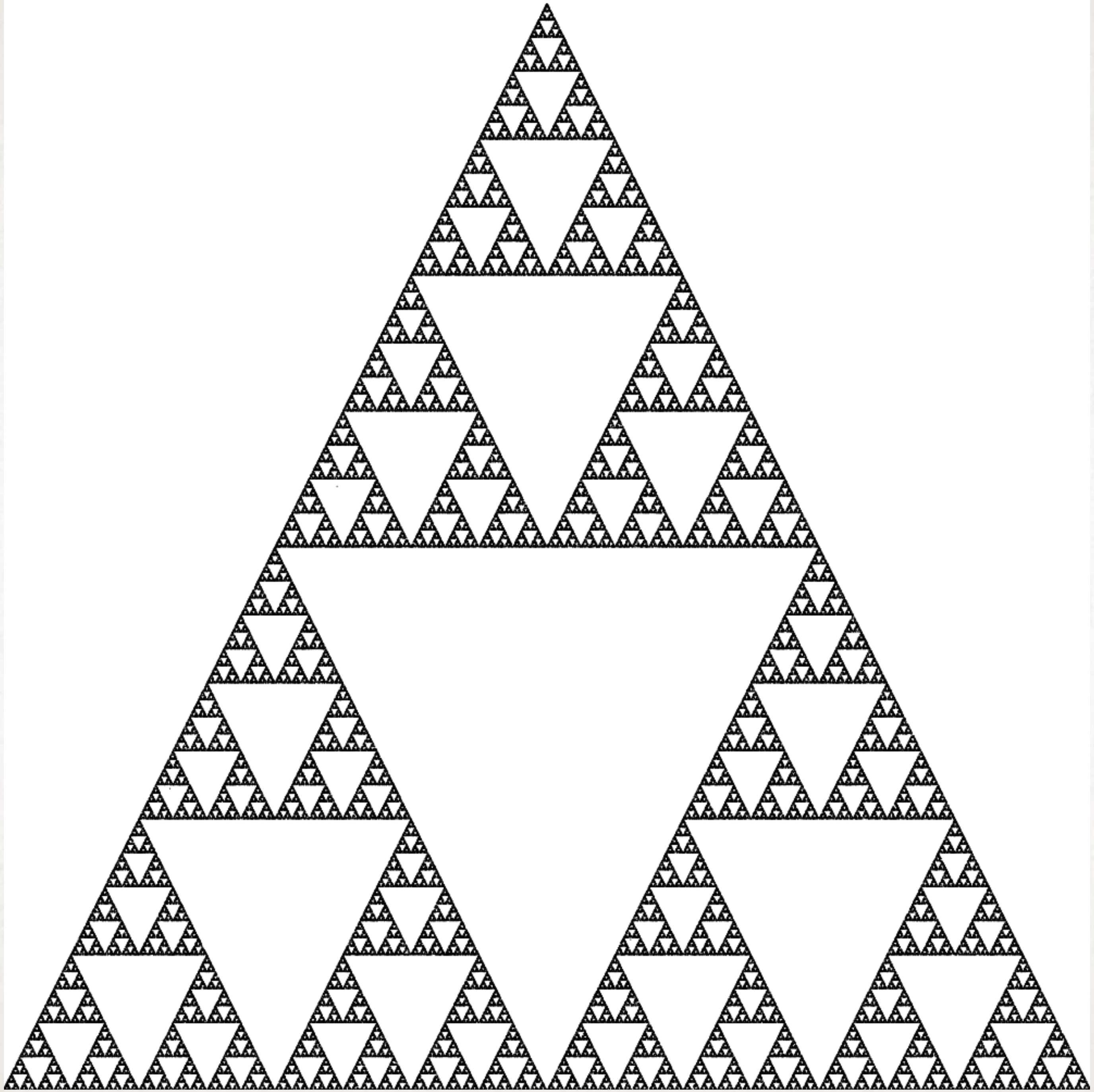
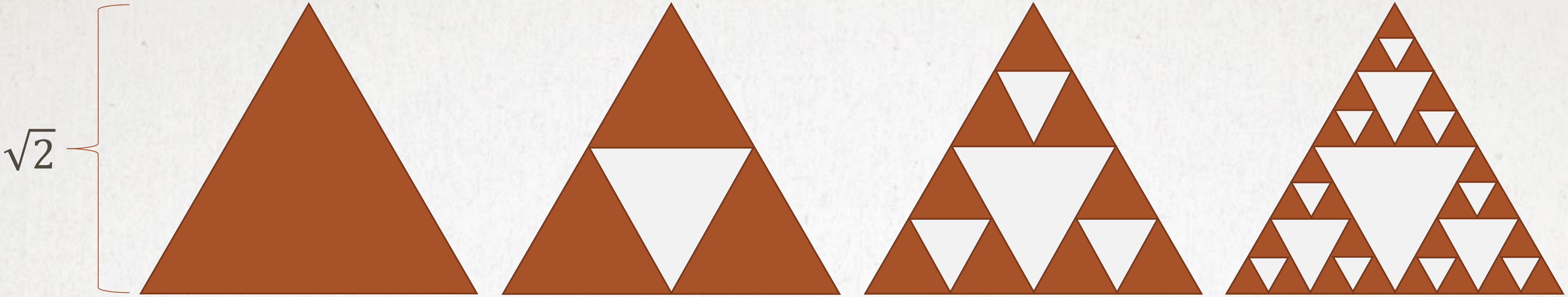


THE SIERPINSKI GASKET



Stage 0:

$$A_0 = \frac{1}{2}\sqrt{2}^2$$

$$A_0 = 1$$

Stage 1:

$$A_1 = 1 - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^2$$

$$= 1 - \frac{1}{4}$$

$$A_1 = \frac{3}{4}$$

Stage 2:

$$A_2 = \frac{3}{4} - \frac{3}{2}\left(\frac{\sqrt{2}}{4}\right)^2$$

$$= \frac{3}{4} - \frac{3}{16} = \frac{9}{16}$$

$$A_2 = \left(\frac{3}{4}\right)^2$$

Stage n:

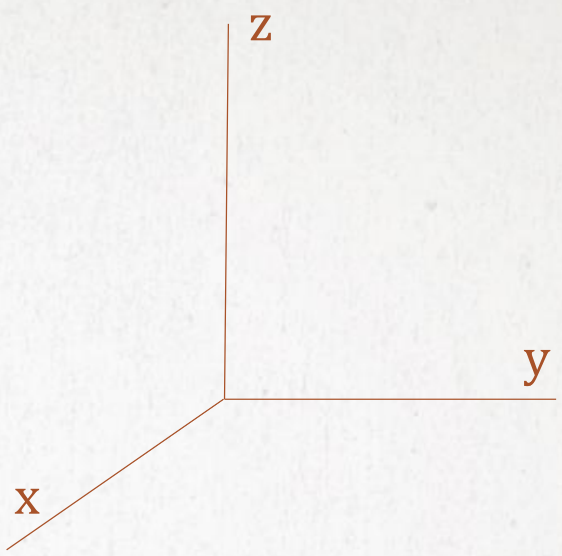
$$A_n = \left(\frac{3}{4}\right)^n$$

$$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$$

Sierpinski Gasket has zero area?

CAPACITY DIMENSION AND FRACTALS

Let $S \subseteq \mathbb{R}^n$ where n = 1, 2, or 3

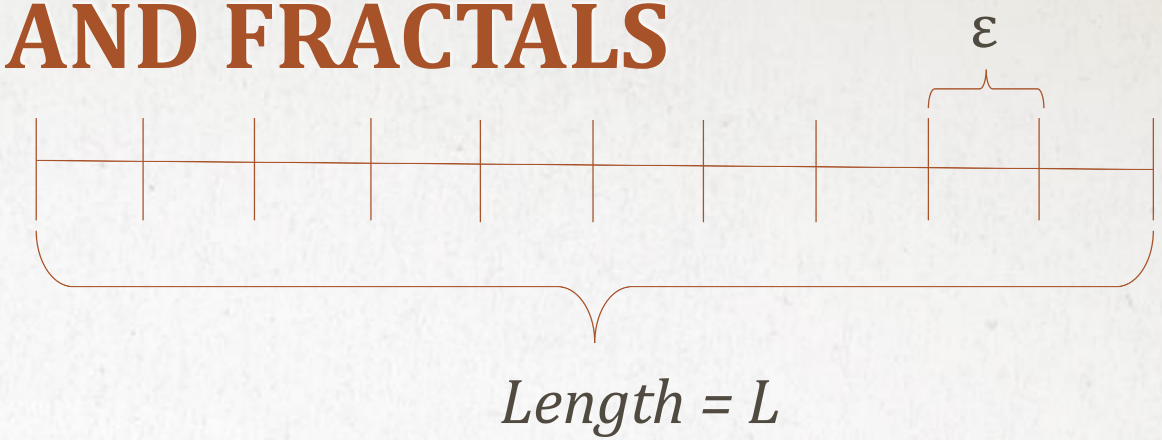


n-dimensional box

- n = 1: Closed interval
- n = 2: Square
- n = 3: Cube

Let N(ε) = smallest number of n-dimensional boxes of side length ε necessary to cover S

CAPACITY DIMENSION AND FRACTALS



n = 1

Boxes of length ε to cover line of length L:

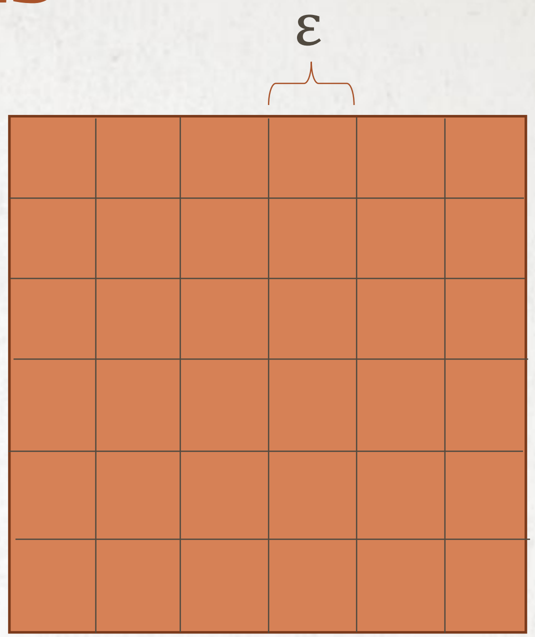
If L = 10cm and ε = 1cm, it takes 10 boxes to cover L

If ε = 0.5cm, it takes 20 boxes to cover L

…

$N(\varepsilon) \propto \frac{1}{\varepsilon}$

CAPACITY DIMENSION AND FRACTALS



$\underline{n = 2}$

Boxes of length ε to cover
square S of side length L

If L = 20cm, area of S = 400cm^2

- ε = 2cm: each box has area 4cm^2
  It will take 100 boxes to cover S
- ε = 1cm: each box has area 1cm^2
  It will take 400 boxes to cover S

$$N(\varepsilon) \propto \frac{1}{\varepsilon^2}$$

CAPACITY DIMENSION AND FRACTALS

$$N(\varepsilon) = C\left(\frac{1}{\varepsilon}\right)^{D}$$

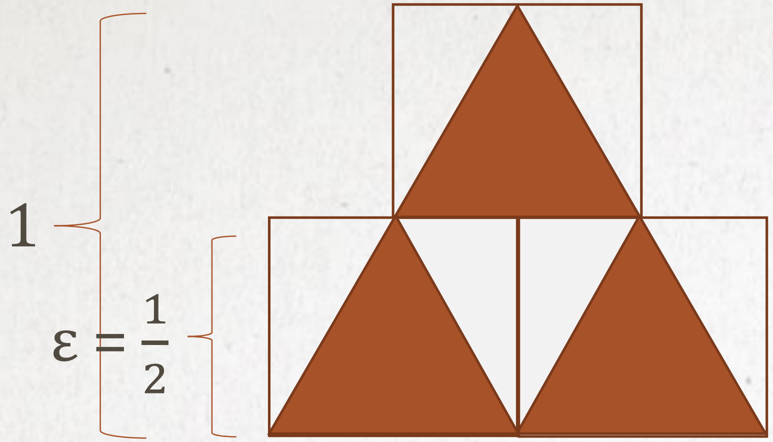
$$\ln\left(N(\varepsilon)\right) = D\ln\left(\frac{1}{\varepsilon}\right) + \ln(C)$$

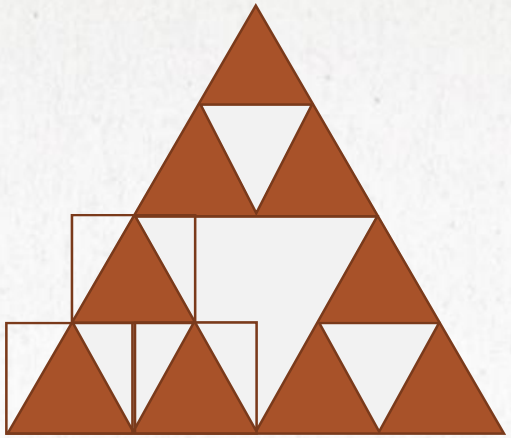
$$D = \frac{\ln\left(N(\varepsilon)\right) - \ln(C)}{\ln\left(\frac{1}{\varepsilon}\right)}$$

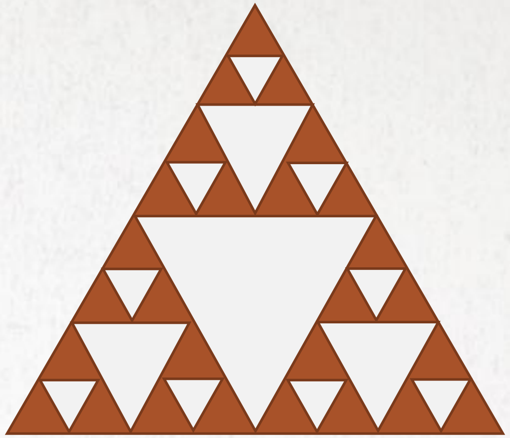
C just depends on scaling of S

Capacity Dimension: $\dim_c S = \lim_{\varepsilon \to 0^+} \frac{\ln\left(N(\varepsilon)\right)}{\ln\left(\frac{1}{\varepsilon}\right)}$

CAPACITY DIMENSION AND FRACTALS







Stage 1:

If $\varepsilon = \frac{1}{2}$, $N(\varepsilon) = 3$

Stage 2:

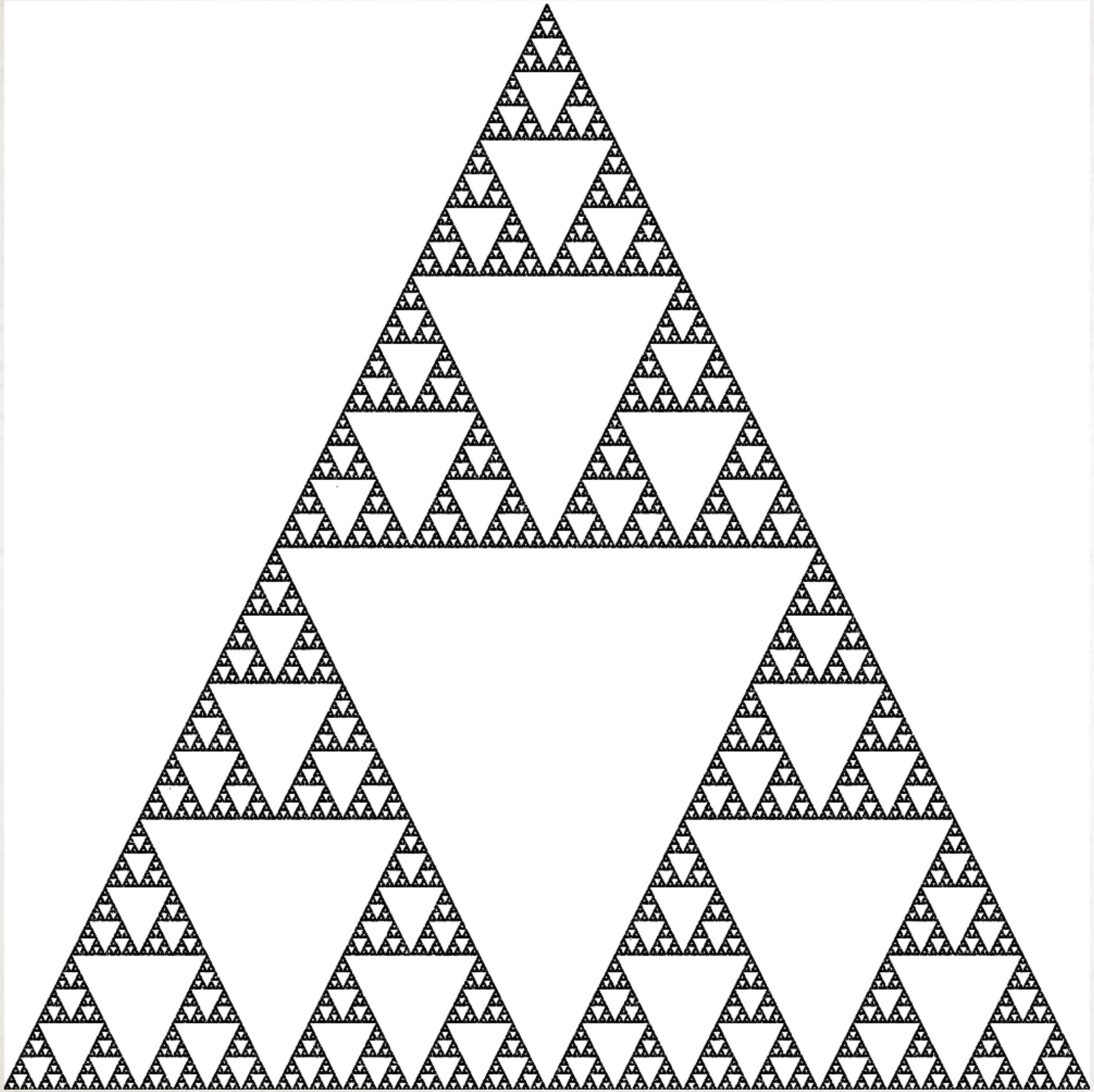
If $\varepsilon = \frac{1}{4}$, $N(\varepsilon) = 9$

Stage n:

If $\varepsilon = \frac{1}{2^n}$, $N(\varepsilon) = 3^n$

$\frac{1}{\varepsilon} = 2^n$

$$\lim_{\varepsilon \to 0^+} \frac{\ln(N(\varepsilon))}{\ln\left(\frac{1}{\varepsilon}\right)} = \lim_{\varepsilon \to 0^+} \frac{\ln(3^n)}{\ln(2^n)} = \frac{n \ln(3)}{n \ln(2)} = \frac{\ln(3)}{\ln(2)} \approx 1.5849625$$



The Sierpinski Gasket is ≈ 1.585 dimensional

A set with non-integer capacity dimension is called a **fractal.**

ITERATED FUNCTION SYSTEMS

An **Iterated Function System** (IFS) *F* is the union of the contractions $T_1, T_2 \dots T_n$

THEOREM: Let *F* be an iterated function system of contractions in $\mathbb{R}^2$. Then there exists a unique compact subset A_F in $\mathbb{R}^2$ such that for any compact set *B*, the sequence of iterates $\{F^n(B)\}_{n=1}^{\infty}$ converges in the Hausdorff metric to A_F

A_F is called the **attractor** of *F*.

This means that if we iterate any compact set in $\mathbb{R}^2$ under *F*, we will obtain a unique attractor (attractor depends on the contractions in F)

ITERATED FUNCTION SYSTEMS

A function T : $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is **affine** if it is in the form

$$f\binom{x}{y} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}\binom{x}{y} + \begin{bmatrix} \mathrm{e} \\ \mathrm{f} \end{bmatrix} = \begin{bmatrix} \mathrm{a}x + \mathrm{b}y + \mathrm{e} \\ \mathrm{c}x + \mathrm{d}y + \mathrm{f} \end{bmatrix}$$

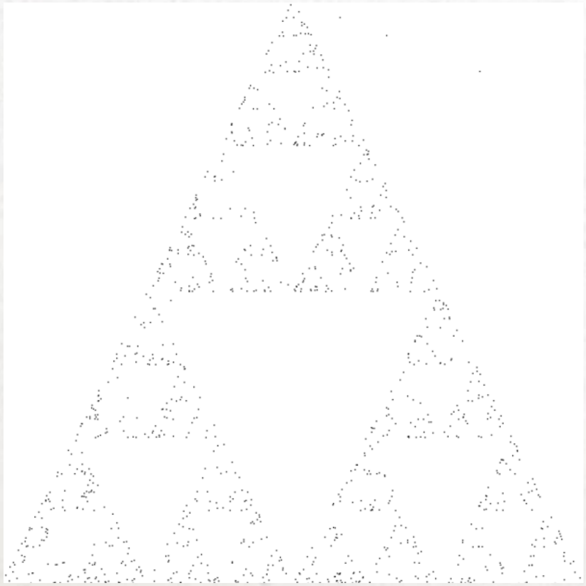
(linear function followed by translation)

We will deal with iterated function systems of affine contractions.

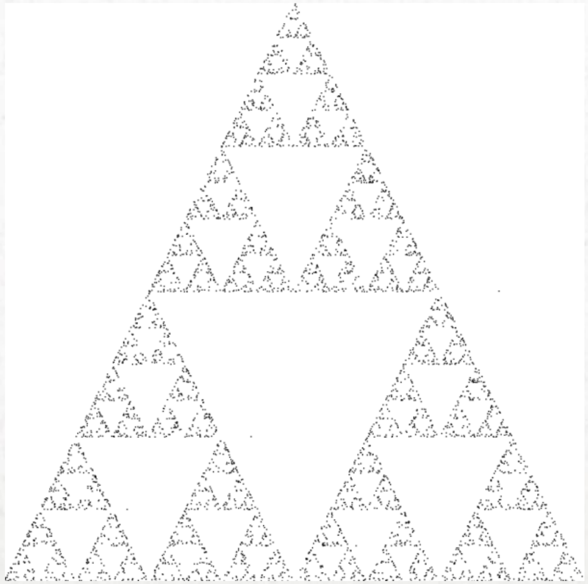
RANDOM ITERATION ALGORITHM

Drawing an attractor of IFS F:

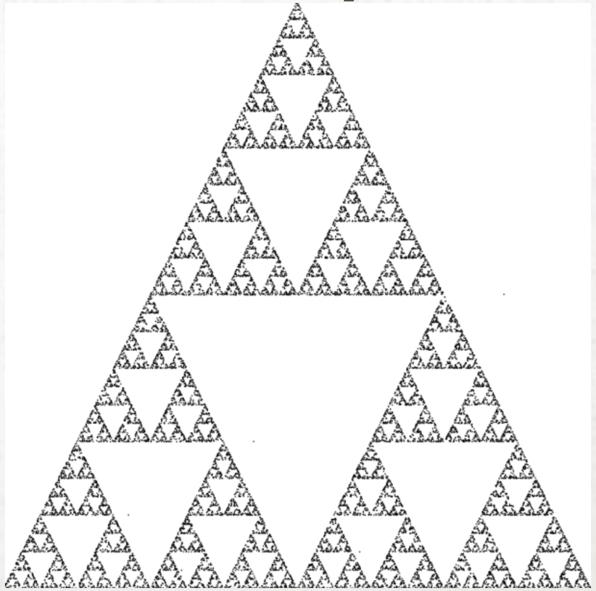
1. Choose an arbitrary initial point $\vec{v} \in \mathbb{R}^2$
2. Randomly select one of the contractions $\mathrm{T_n}$ in F
3. Plot the point $\mathrm{T_n}(\vec{v})$
4. Let $\mathrm{T_n}(\vec{v})$ be the new $\vec{v}$
5. Repeat steps 2-4 to obtain a representation of A_F



1000 Iterations



5000 Iterations



20,000 Iterations

LET'S DRAW SOME FRACTALS!

ITERATIONS AND FIXED POINTS

Iterating = repeating the same procedure

Let $f(x)$ be a function.

$f\left(f(x)\right) = f^2(x)$ is the second iterate of x under f.

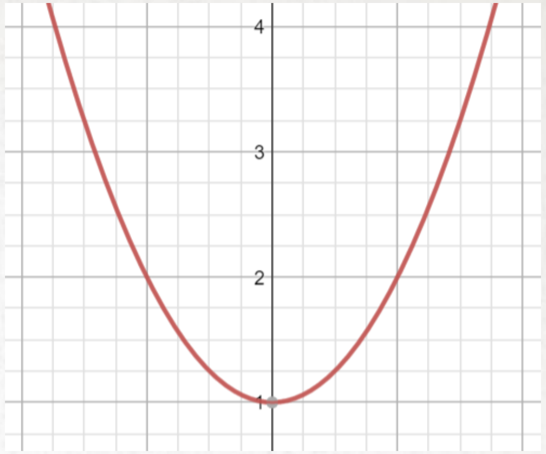
$f(f(f((x))) = f^3(x)$ is the third iterate of x under f

Example: $f(x) = x^2 + 1$

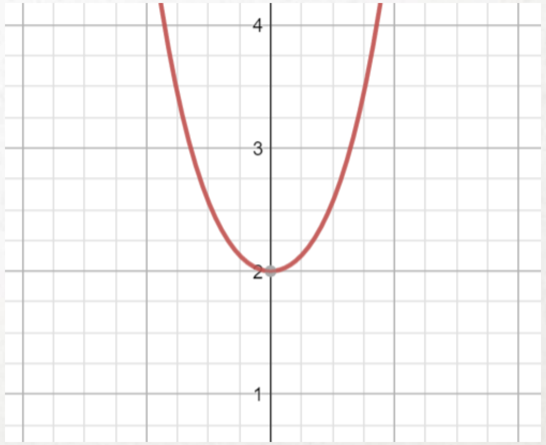
$f(5) = 26$

$f\left(f(5)\right) = f^2(5) = f(26) = 677$

$f(x) = x^2 + 1$



$f^2(x) = (x^2+1)^2 + 1$

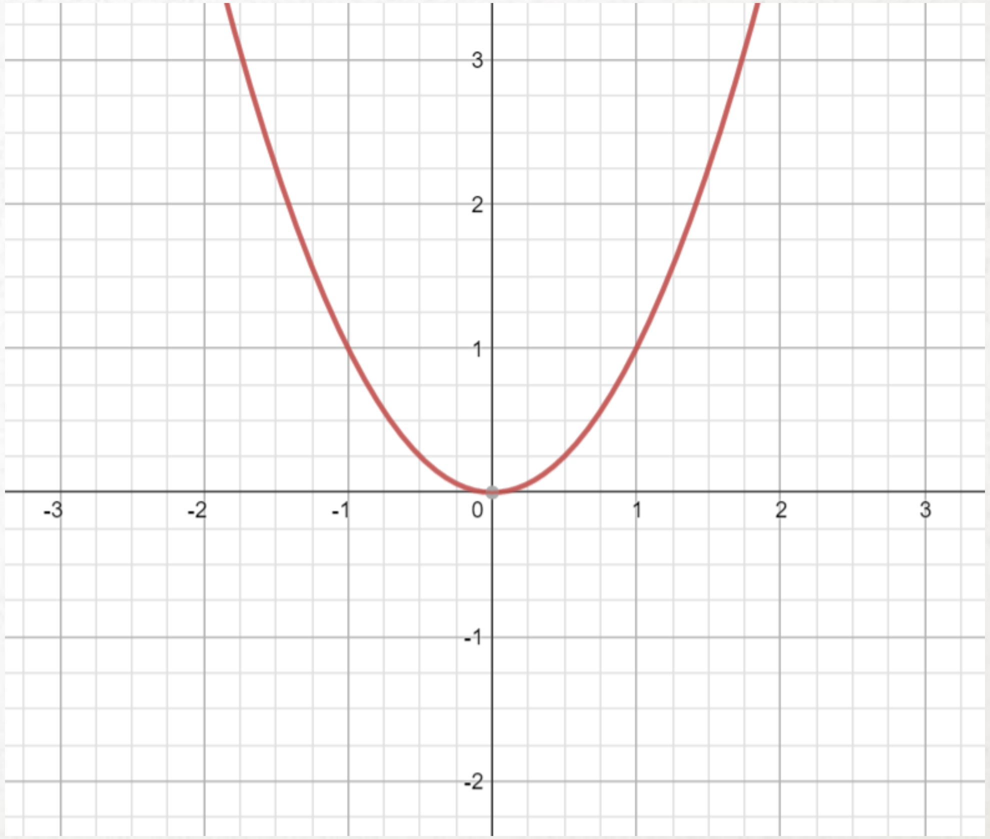


ITERATIONS AND FIXED POINTS

A point p is a **fixed point** for a function f if its iterate is itself

$$f(p) = p$$

$f(x) = x^2$



Example: $f(x) = x^2$

$$f(0) = 0$$

$$f\big(f(0)\big) = f^2(0) = f(0) = 0$$

Therefore 0 is a fixed point of f

METRIC SPACES

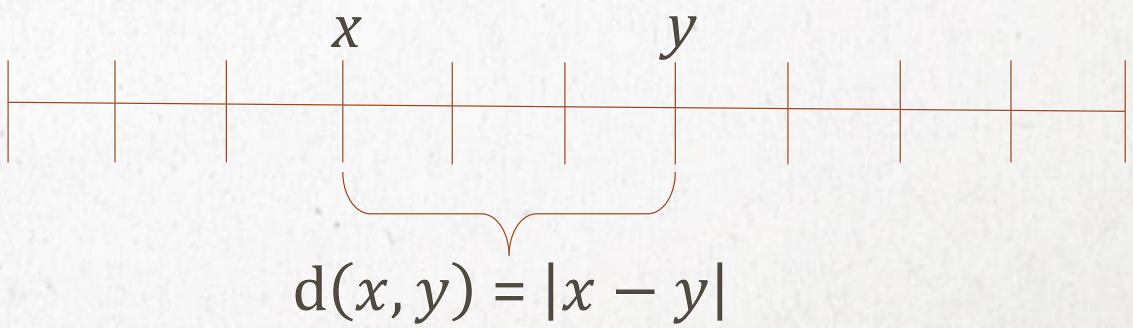
Let S be a set. A **metric** is a distance function $\mathrm{d}(x, y)$ that satisfies 4 axioms $\forall\, x, y \in S$

1. $\mathrm{d}(x, y) \geq 0$
2. $\mathrm{d}(x, y) = 0$ if and only if $x = y$
3. $\mathrm{d}(x, y) = \mathrm{d}(y, x)$
4. $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$

Example: Absolute Value $\mathbb{R}$

$\mathrm{d}(x, y) = |x - y|$

$(\mathbb{R}, \mathrm{d})$ is a **metric space**



METRIC SPACES

Let (S, d) be a metric space.

A sequence $\{x_n\}_{n=1}^{\infty}$ in S **converges** to $x \in \mathrm{S}$ if $\lim_{n\to\infty} \mathrm{d}(x_n, x) = 0$

This means that terms of the sequence approach a value s

A sequence is **Cauchy** if for all $\varepsilon > 0$ there exists a positive integer N such that whenever $n, m \geq N$, $\mathrm{d}(x_n, x_m) < \varepsilon$

This means that terms of the sequence get closer together

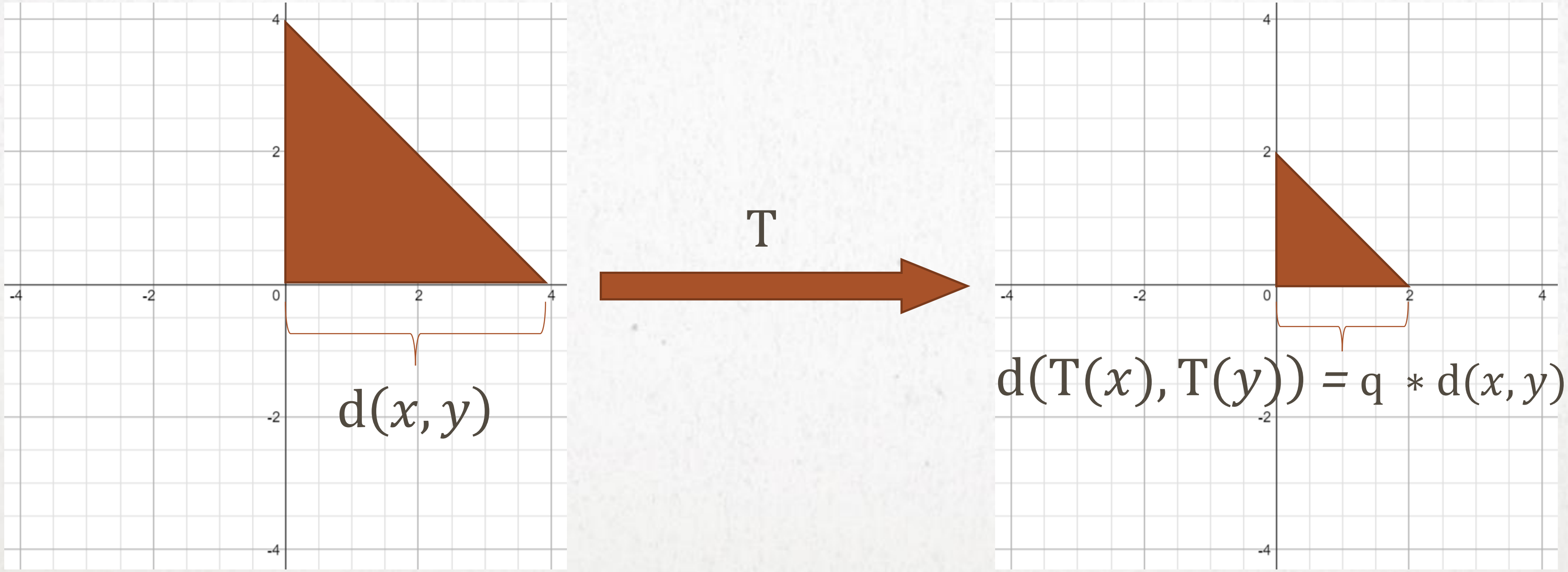
(S, d) is a **complete metric space** if every Cauchy sequence in S converges to a member of S

CONTRACTION MAPPING THEOREM

Let (S, d) be a metric space.

A function $T: S \to S$ is a **contraction** if $\exists\ q \in [0,1)$ such that

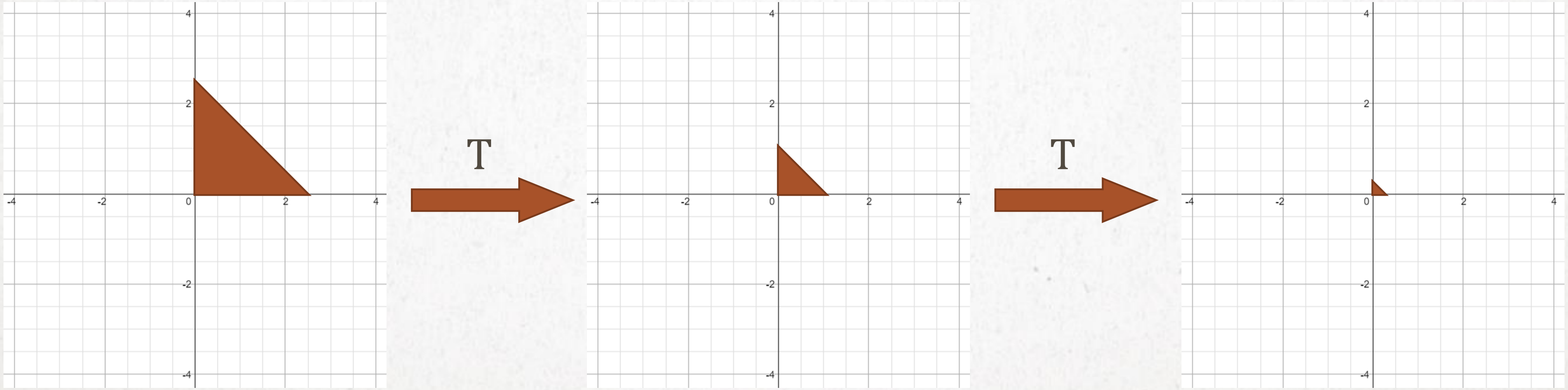
$$d(T(x), T(y)) \leq q * d(x, y)$$



CONTRACTION MAPPING THEOREM

Contraction Mapping Theorem:

If (S, d) is a complete metric space, and T is a contraction, then as n $\rightarrow \infty, T^n(x) \rightarrow$ unique fixed point $x^* \; \forall \; x \in S$



HAUSDORFF METRIC

A set S is **closed** if whenever x is the limit of a sequence of members of T, x actually is in T.

A set S is **bounded** if it there exists $x \in$ S and $r > 0$ such that $\forall\ s \in$ S, d(x, s) < r

Means S is contained by a "ball" of finite radius

A set S $\subseteq \mathbb{R}^n$ is **compact** if it is closed and bounded

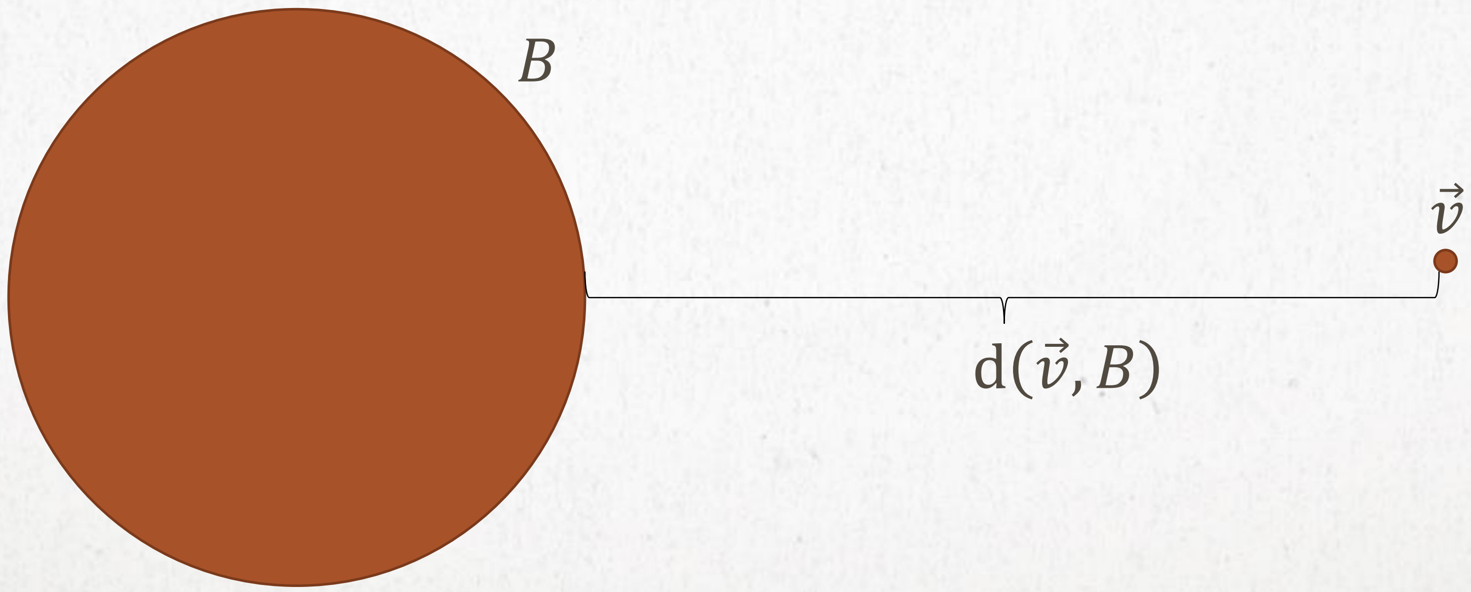
Let K denote all compact subsets of $\mathbb{R}^2$

HAUSDORFF METRIC

If B is a nonempty member of K, and $\vec{v}$ is any point in $\mathbb{R}^2$, the distance from $\vec{v}$ to B *is*

$$d(\vec{v}, B) = \text{minimum value of } \left\|\vec{v} - \vec{b}\right\| \ \forall\, \vec{b} \in B$$

(distance from point to a compact set)



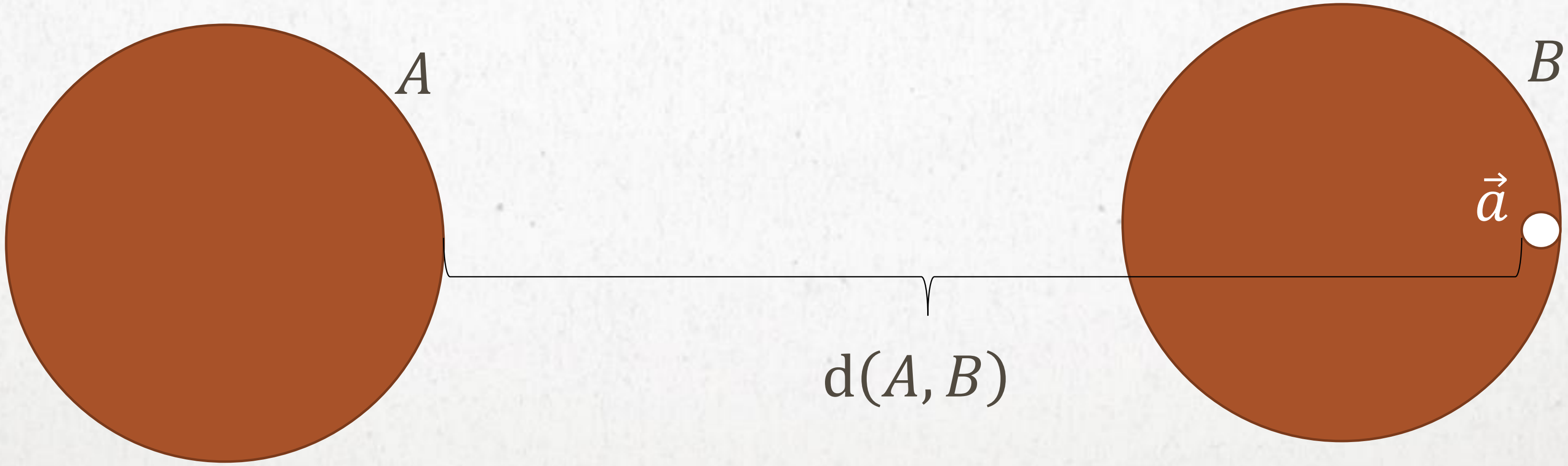
HAUSDORFF METRIC

If A and B are members of K, then the distance from A to B is

$$\mathrm{d}(A, B) = \text{maximum value of } \mathrm{d}(\vec{a}, B) \ \text{ for } \vec{a} \in A$$

(distance between compact sets)

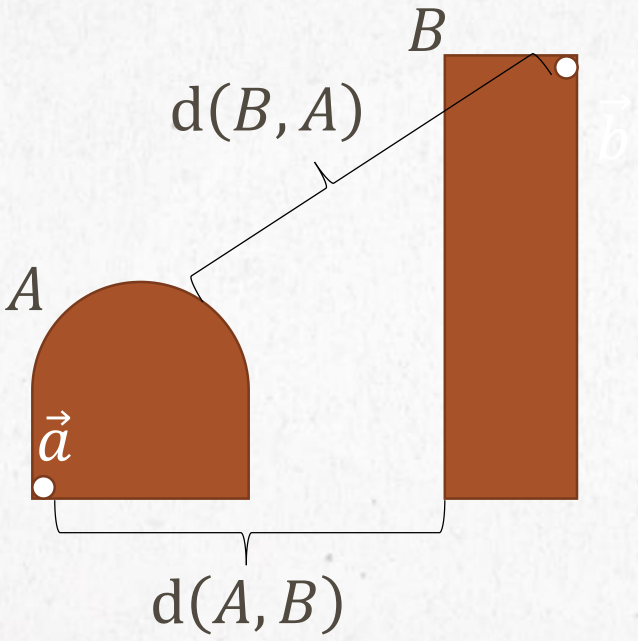
Means we take the point in A that is most distant from any point in B and find the minimum distance between it an any point in B



HAUSDORFF METRIC

The **Hausdorff metric** on K is defined as:

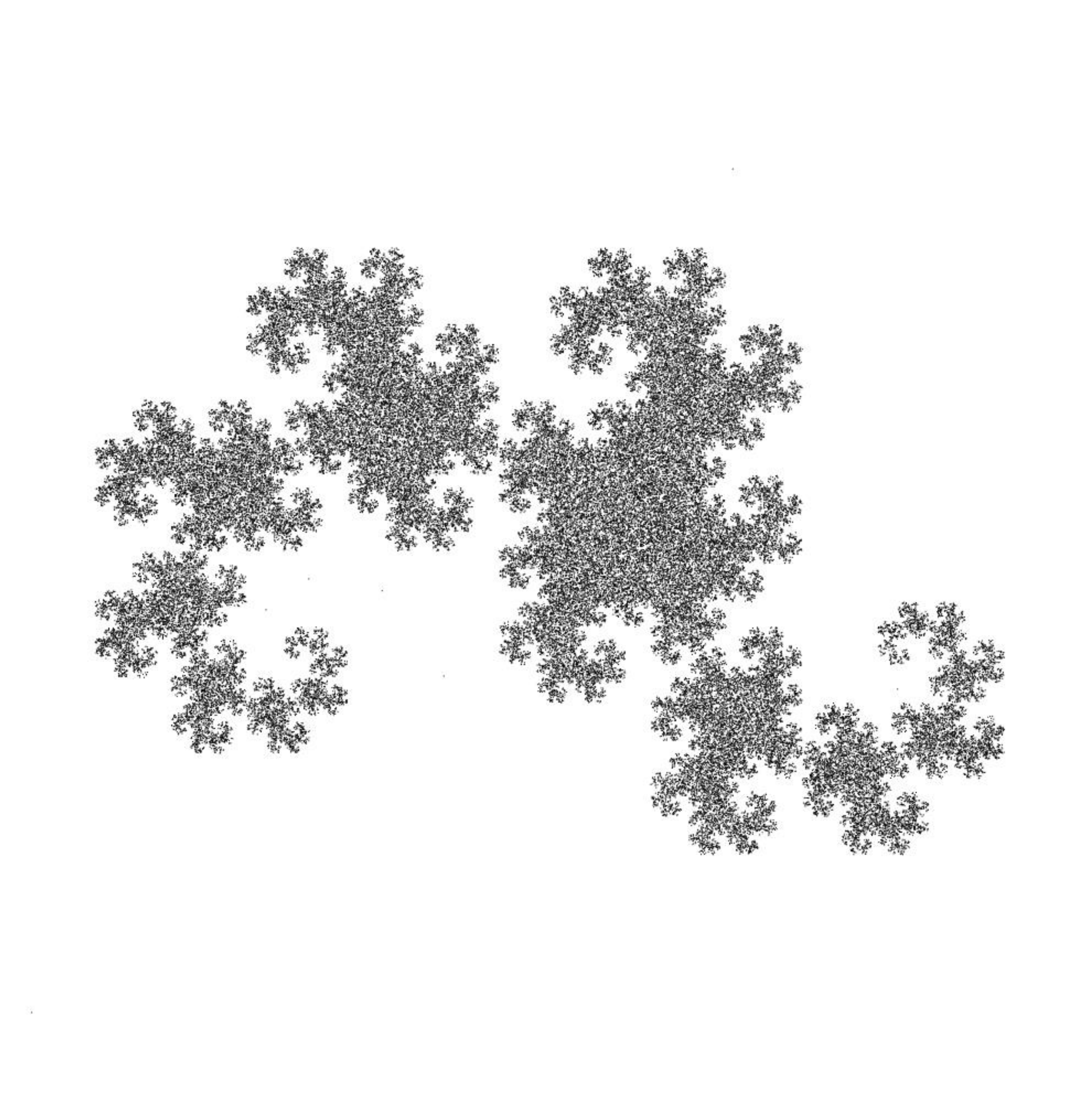
$$\mathrm{D}(A, B) = \text{maximum of } \mathrm{d}(A, B) \text{ and } \mathrm{d}(B, A)$$



$\mathrm{d}(A, B)$

$\mathrm{d}(B, A)$

$$\mathrm{D}(A, B) = \mathrm{d}(B, A)$$



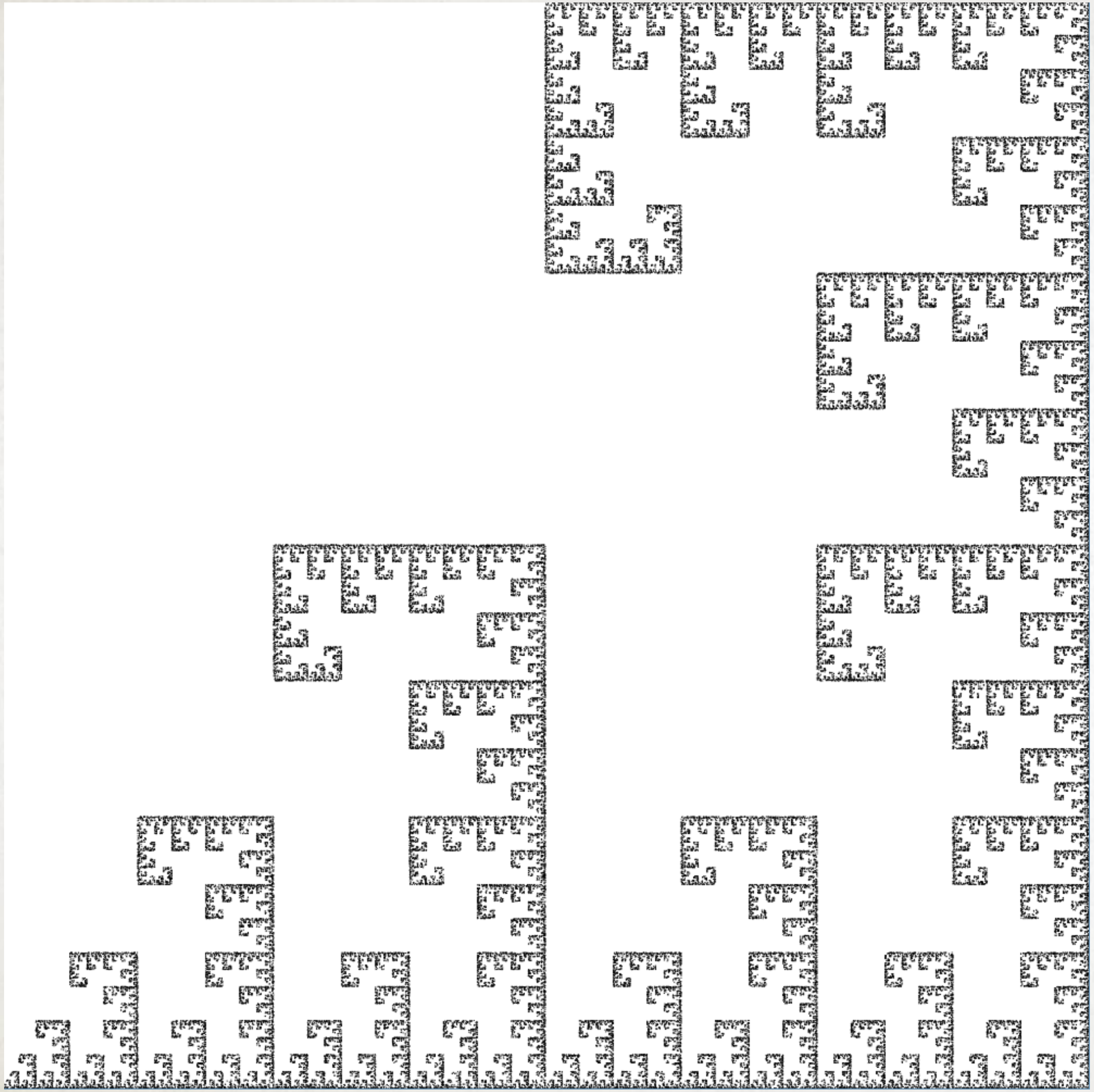
Iterated Function System:

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1/2 & -1/2\\1/2 & 1/2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$f_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}-1/2 & -1/2\\1/2 & -1/2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$$

Render Details:

Point size: 1px
# of iterations: 100,000



Iterated Function System:

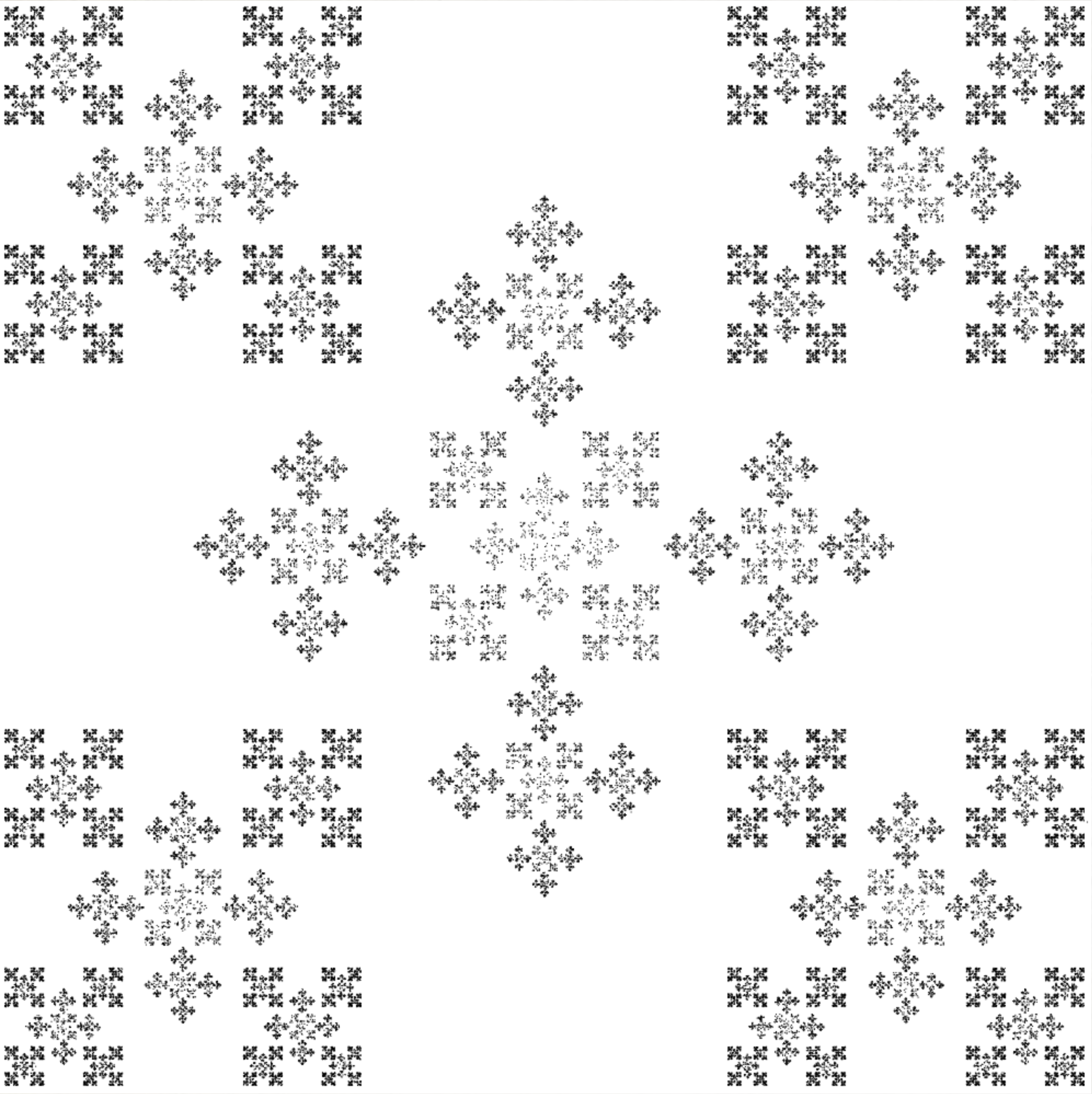
$$f_1\binom{x}{y} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\binom{x}{y}$$

$$f_2\binom{x}{y} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\binom{x}{y} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$f_3\binom{x}{y} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}\binom{x}{y} + \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$$

Render Details:

Point size: 1px
# of iterations: 100,000



Iterated Function System:

$$f_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{bmatrix}2/3\\2/3\end{bmatrix}$$

$$f_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{bmatrix}2/3\\-2/3\end{bmatrix}$$

$$f_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{bmatrix}-2/3\\2/3\end{bmatrix}$$

$$f_4\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{bmatrix}-2/3\\-2/3\end{bmatrix}$$

$$f_5\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}13/40 & 13/40\\ -13/40 & 13/40\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Render Details:

Point size: 1px

# of iterations: 100,000

Iterated Function System:

$$f_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}0 & 0\\0 & .16\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

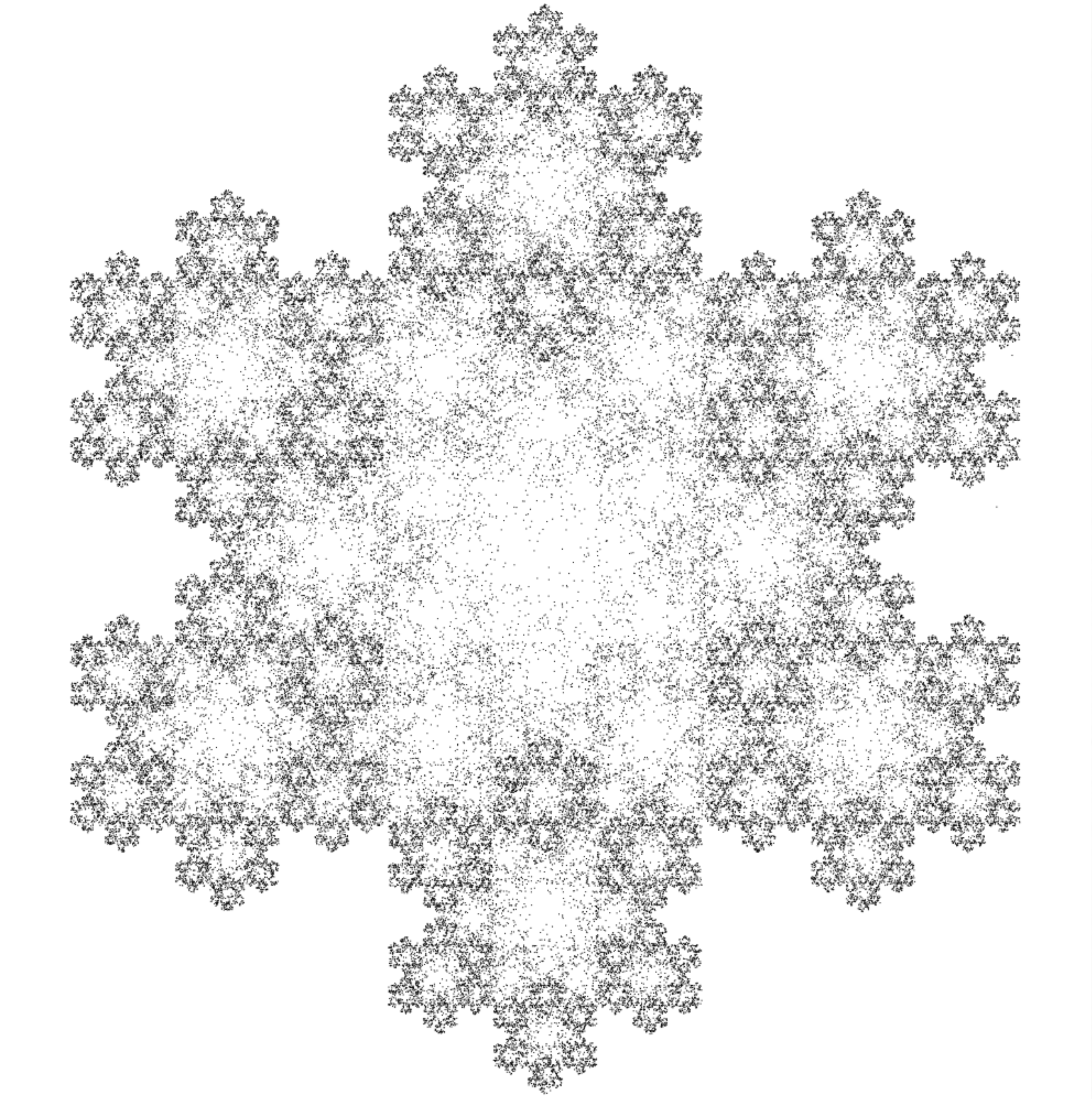
$$f_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}.85 & .04\\-.04 & .85\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}0\\1.6\end{bmatrix}$$

$$f_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}.2 & -.26\\.23 & .22\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}0\\1.6\end{bmatrix}$$

$$f_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}-.15 & .28\\.26 & .24\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}0\\.44\end{bmatrix}$$

Probabilities:

f_1: 1% f_2: 85%
f_3: 7% f_4: 7%



Iterated Function System:

$$f_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/2 & -\sqrt{3}/6\\ \sqrt{3}/6 & 1/2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$f_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}1/\sqrt{3}\\ 1/3\end{bmatrix}$$

$$f_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}1/\sqrt{3}\\ -1/3\end{bmatrix}$$

$$f_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}-1/\sqrt{3}\\ 1/3\end{bmatrix}$$

$$f_5\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}-1/\sqrt{3}\\ -1/3\end{bmatrix}$$

$$f_6\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}0\\ 2/3\end{bmatrix}$$

$$f_7\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}1/3 & 0\\ 0 & 1/3\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{bmatrix}0\\ -2/3\end{bmatrix}$$

THANKS

Mentor – Kasun Fernando

Author and Book Provider – Denny Gulick

Some IFS Formulas from Agnes Scott College

Graphs Created with Desmos Graphing Calculator